Traffic in Connecting Networks When Existing Calls Are Rearranged

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Recent mathematical[1,2] and engineering studies[3,4] of the possibility of rearranging existing calls in a connecting network (so as to add more calls) have raised the problem of loss (probability of blocking) for networks operated in this manner. We consider and solve this problem in the context of a Markov traffic model, for a connecting network operated according to the rule that if a new call is blocked, but can be accommodated by rearranging the calls in progress, then it is put up, after some choice of rearrangement.

I. INTRODUCTION

Each state of the network realizes a given assignment of inlets to outlets, a specification of who is to talk to whom. There is a natural map γ that takes the set S of states into the set A of assignments. The equilibrium probabilities of the process x_t on S we use as a traffic model are complicated and unknown functions of the offered load. But it turns out that, because of the policy of rearranging whenever necessary, $\gamma(x_t)$ is also a Markov process, and one whose state probabilities are easy to calculate.

We give explicit analytical formulas for the equilibrium probability of a given assignment of inlets to outlets, for the probability of n calls in progress. For networks that are one-sided (inlets = outlets) or two-sided (inlets $\cap$ outlets = φ), all the important constants of traffic engineering can be obtained from a *partition function*, a polynomial in the offered load with coefficients depending only on network structure; these constants are the loss, the load carried, the calling rate, and the load variance. These analytical formulas arise from an unexpected connection with the "thermodynamic" model for telephone traffic, described in an earlier paper. As an application, we solve the problem of calculating the loss in a connecting network, made of stages of

rectangular switches, which is rearrangeable except for the fact that the outermost stages are concentrators.

II. STATES AND ASSIGNMENTS

A mathematical model like that of Ref. 5 will be used. The elements of this model separate naturally into combinatorial and probabilistic ones. The former arise from the structure of the connecting network and from the ways in which calls can be put up in it; the latter represent assumptions about the random traffic the network is to carry. We discuss the combinatorial and structural aspects in this section; terminology and notation for these aspects are introduced. The probabilistic aspects are considered in a later section.

A connecting network ν is a quadruple $\nu = (G, I, \Omega, S)$, where G is a graph depicting network structure, I is the set of nodes of G which are inlets, Ω is the set of nodes of G that are outlets, and S is the set of permitted states.[1] Variables x, y, and z at the end of the alphabet denote states, while u and v (respectively) denote a typical inlet and a typical outlet. A state x can be thought of as a set of disjoint chains on G, each chain joining I to Ω. Not every such set of chains represents a state: sets with wastefully circuitous chains may be excluded from S. It is possible that $I = \Omega$, that $I \cap \Omega = \varphi =$ null set, or that some intermediate condition obtains, depending on the "community of interest" aspects of the network ν.

The set S of states is partially-ordered by inclusion $\leqq$, where $x \leqq y$ means that state x can be obtained from state y by removing zero or more calls. If x and y satisfy the same assignment of inlets to outlets—that is, are such that all and only those inlets $u \in I$ are connected in x to outlets $v \in \Omega$ which are connected to the same v in y (though possibly by different routes), then we say that x and y are equivalent, written $x \sim y$.

We denote by A_x the set of states that are immediately above x in the partial ordering $\leqq$, and by B_x the set of those that are immediately below. Thus

$$A_x = \{\text{states accessible from } x \text{ by adding a call}\}$$

$$B_x = \{\text{states accessible from } x \text{ by a hangup}\}.$$

Also, we let A_{cx} be the set of states that could result from x by putting up call c. A call c is new in x if the terminals of c are idle in x. A call c new in x is *blocked* in x if there is no $y \in A_x$ with c in progress in y; it is

completely blocked in x if there is no way of rearranging the calls of x to an equivalent state z, $z \sim x$, such that c is not blocked in z.

The number of calls in progress in state x is denoted by $|x|$. The number of calls which are not completely blocked in x is denoted by $\sigma(x)$, for "successes in x." The functions $|\cdot|$ and $\sigma(\cdot)$ defined on S play important roles in the stochastic process to be used for studying traffic. In addition, we use the notations

β_x = number of idle inlet-outlet pairs completely blocked in state x,

α_x = number of idle inlet-outlet pairs in state x,

and we note that $\alpha = \beta + \sigma$.

It can be seen, further, that the set S of states is not merely partially ordered by $\leqq$, but also forms a semi-lattice, or a partially ordered system with intersections, with $x \cap y$ defined to be the state consisting of those calls and their respective routes which are common to both x and y.

An assignment specifies what inlets should be connected to what outlets. The set A of assignments can be represented as the set of all fixed-point-free correspondences from subsets of I to Ω. The set A is partially ordered by inclusion, and there is a natural map $\gamma(\cdot): S \to A$ which takes each state $x \varepsilon S$ into the assignment it realizes; the map $\gamma(\cdot)$ is a semilattice homomorphism of S into A, since

$$x \geqq y \text{ implies } \gamma(x) \geqq \gamma(y),$$

$$\gamma(x \cap y) \leqq \gamma(x) \cap \gamma(y).$$

Variables a and b are used for members of A. For $a \varepsilon A$, $|a|$ is the number of inlets (or outlets) which are "busy" if assignment a is specified.

A *unit* assignment is, naturally, one that assigns exactly one inlet to some one outlet, and it corresponds to having just one call in progress. It is convenient to identify new calls c and unit assignments, and to write $\gamma(x) \cup c$ for the larger assignment consisting of $\gamma(x)$ and the call c together, with the understanding of course that none of the terminals of c is busy in $\gamma(x)$.

Remark 1: Not every assignment need be realizable by some state of S. Indeed, it is common for practical networks to realize only a small fraction of the possible assignments. Since we are studying a network operated with rearranging when necessary, blocking will occur in a state x only when a call c idle in x is completely blocked in the

sense that no state realizes $\gamma(x) \cup c$; that is, when $\gamma(x) \cup c \not\in \gamma(S)$.

The set $\gamma(S)$ of realizable assignments is also partially ordered by inclusion. We use the notations, for $a \in \gamma(S)$,

$$A_a = \{\text{realizable assignments immediately above } a\}$$

$$B_a = \{\text{realizable assignments just below } a\}.$$

$| X |$ is the number of members of a set X.

Remark 2: For $x \in S$, $\sigma(x) = | A_{\gamma(x)} |$.

The set of calls which can be put up in state y is the same for all $y \in \gamma^{-1}[\gamma(x)]$.

III. ASSUMPTIONS

A Markov stochastic process x_t taking values on S is used as a mathematical description of an operating connecting network subject to random traffic. A Markov process similar to that of Ref. 1 will be used. This model can be paraphrased in the informal terminology of "rates" by two simple assumptions:

(*i*) The hang-up rate per call in progress is unity.

(*ii*) The calling-rate between an inlet and a distinct outlet, both idle at time u, is $\lambda > 0$.

The transition probabilities of x_t will be described after a discussion of system operation and routing.

It will be assumed that attempted calls to busy terminals are rejected, and have no effect on the state of the network. Successful attempts to place a call are completed instantly with some choice of route, or are rejected, in accordance with some *routing policy*.

It remains to say what happens to blocked or unsuccessful attempts to make a call. We assume that a policy of total rearranging is followed, according to which if a call can be put up at all, by a rearrangement of the existing calls if necessary, then it is completed. We say a call c is *completely blocked* in x if there is no state whatever satisfying the assignment $\gamma(x) \cup c$, that is, if there is no way of rearranging the calls in progress in x so as to create a free path for c. Completely blocked calls are refused, with no change of state.

The map γ is fundamental in our study of the effect of rearranging on blocking. This is because if c is a new call blocked in x which is accommodated by rearranging, then the resulting state of the system is one of the states in $\gamma^{-1}\{\gamma(x) \cup c\}$.

It is convenient to modify the matrix R, used in Refs. 1 and 5, to

cover rearranging as well as routing. This will be done by adding some more non-zero entries to represent ways of putting up calls by network rearrangement, for example, calls that are blocked but not completely blocked. A general description will be given which actually covers the use of rearrangement for unblocked calls.

Consider the set $\gamma^{-1}\{A_{\gamma(x)}\}$, consisting of all states obtainable from x by (possibly) rearranging the calls in x and then adding another call. The equivalence relation $\sim$ of "having the same calls up" (or satisfying the same assignment of inlets to outlets) induces a partition Π_x of $\gamma^{-1}\{A_{\gamma(x)}\}$. It can be seen that Π_x consists of exactly the sets $\gamma^{-1}\{\gamma(x) \cup c\}$ for calls c not completely blocked in x. For $Y \in \Pi_x$, r_{xy} for $y \in Y$ is a probability distribution over Y, and $r_{xy} = 0$ in all other cases.

The interpretation of the matrix R is to be this: any $Y \in \Pi_x$ represents all the ways in which a call c (free and not completely blocked in x) could be added when the network state is x, both with and without rearranging; for $y \in Y$, r_{xy} is the chance that if call c is attempted in state x, it will be completed (by choice of a route for c and possibly by choice of a rearrangement for x) so as to take the system to state y.

It is to be noted that routing and rearranging are carried out with complete knowledge about the current state of the network.

IV. PRINCIPAL RESULTS

The probabilistic and operational assumptions we have made give rise to a Markov stochastic process x_t taking values on S. This process is determined by its transition rate matrix $Q = (q_{xy})$, given by

$$q_{xy} = \begin{cases} 1 & y \in B_x \\ \lambda r_{xy} & y \in \gamma^{-1}(A_{\gamma(x)}) \\ -|x| - \lambda\sigma(x) & y = x \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1: $\gamma(x_t)$ is a Markov process with transition rate matrix $\Gamma = (\gamma_{ab})$ given by

$$\gamma_{ab} = \begin{cases} 1 & b \in B_a \\ \lambda & b \in A_a \\ -|a| - \lambda\,|A_a| & b = a \\ 0 & \textit{otherwise.} \end{cases}$$

Proof: Since x_t is itself a Markov process, it is enough to prove that if $0 < t_1 < \cdots < t_n$ then

$$\operatorname{distr}\{\gamma(x_{t_1}), \cdots, \gamma(x_{t_n}) \mid x_0\}$$

depends only on $\gamma(x_0)$. We have

$$P\{\gamma(x_{t_i}) = a_i\,, \quad i = 1, \cdots, n \mid x_0\} = \sum_{x_i \varepsilon \gamma^{-1}(a_i)} P\{x_{t_i} = x_i\,, \quad i = 1, \cdots, n \mid x_0\};$$

with $p_{xy}(t) = P\{x_t = y \mid x_0 = x\}$, this is

$$\sum_{x_1 \varepsilon \gamma^{-1}(a_1)} \cdots \sum_{x_n \varepsilon \gamma^{-1}(a_n)} \prod_{i=1}^{n} p_{x_{i-1}x_i}(t_i - t_{i-1}) = \sum_{x_1 \varepsilon \gamma^{-1}(a_1)} \cdots \sum_{x_{n-1} \varepsilon \gamma^{-1}(a_{n-1})} \cdot \prod_{i=1}^{n-1} p_{x_{i-1}x_i}(t_i - t_{i-1}) \sum_{x_n \varepsilon \gamma^{-1}(a_n)} p_{x_{n-1}x_n}(t_n - t_{n-1}).$$

If we can show that for $x, a \;\varepsilon\; S \times A$, the function $\varphi_{xa}(\cdot)$ defined by

$$\varphi_{xa}(t) = \sum_{y \varepsilon \gamma^{-1}(a)} p_{xy}(t)$$

depends only on $\gamma(x)$, the rightmost sum above will factor out because it would depend only on a_{n-1}. By iteration it would follow that with z_i an arbitrary element of $\gamma^{-1}(a_i)$

$$P\{\gamma(x_{t_i}) = a_i, \quad i = 1, \cdots, n \mid x_0\} = \prod_{i=1}^{n} \sum_{y \varepsilon \gamma^{-1}(a_i)} p_{z_i y}(t_i - t_{i-1}),$$

whence the theorem.

Tbe matrix function $P(t) = [p_{xy}(t)]$ satisfies the *backward* Kolmogorov equation

$$\frac{d}{dt} p_{xy}(t) = \sum_{z \varepsilon B_x} p_{zy}(t) + \lambda \sum_{z \varepsilon \gamma^{-1}(A_{\gamma(x)})} r_{xz} p_{zy}(t) - [|x| + \sigma(x)] p_{xy}(t).$$

By integration, this is equivalent to the integral equation

$$p_{xy}(t) = e^{-[|x| + \lambda\sigma(x)]t} \delta_{xy} + \int_0^t e^{-[|x| + \lambda\sigma(x)]u} \left[\sum_{z \varepsilon B_x} p_{zy}(t - u) + \lambda \sum_{z \varepsilon \gamma^{-1}(A_{\gamma(x)})} r_{xz} p_{zy}(t - u) \right] du.$$

Let $a \;\varepsilon\; \gamma(S)$, and sum over $y \;\varepsilon\; \gamma^{-1}(a)$ to find, with $q_x = |x| + \lambda\sigma(x)$,

$$\varphi_{xa}(t) = e^{-q_x t} \delta_{\gamma(x)a} + \int_0^t e^{-q_x u} \left[\sum_{z \varepsilon B_x} \varphi_{za}(t - u) + \lambda \sum_{z \varepsilon \gamma^{-1}(A_{\gamma(x)})} r_{xz} \varphi_{za}(t - u) \right] du.$$

These equations can be solved by successive approximations as

$$\varphi_{xa}(t) = \lim_{n\to\infty} \psi_{xa}^{(n)}(t)$$

with $\psi_{xa}^{(0)}(t) = e^{-q_x(t)}\,\delta_{\gamma(x)a}$ and

$$\psi_{xa}^{(n+1)}(t) = \psi_{xa}^{(0)}(t) + \int_0^t e^{-q_x u}\left[\sum_{z\varepsilon B_x} \psi_{za}^{(n)}(t-u) + \lambda \sum_{z\varepsilon\gamma^{-1}(A_{\gamma(x)})} r_{xz}\psi_{za}^{(n)}(t-u)\right] du.$$

We have $q_x = |x| + \lambda\sigma(x) = |\gamma(x)| + \lambda |A_{\gamma(x)}|$, by Remark 1, so $\psi_{xa}^{(0)}$ depends only on $\gamma(x)$. As a hypothesis of induction, suppose that $\psi_{za}^{(n)}$ depends only on $\gamma(z)$. Then, since $z \varepsilon B_x$ implies $\gamma(z) \varepsilon B_{\gamma(x)}$ and $|B_x| = |B_{\gamma(x)}| = |x|$, with $\psi_{ba}^{(n)}(t)$ the value of $\psi_{za}^{(n)}(t)$ on $\gamma^{-1}(b)$,

$$\sum_{z\varepsilon B_x} \psi_{za}^{(n)}(t-u) = \sum_{b\varepsilon B\gamma(x)} \psi_{ba}^{(n)}(t-u)$$

depends only on $\gamma(x)$.

Similarly, since $\sum_{z\varepsilon\gamma^{-1}(b)} r_{xz} = 1$ if $b \varepsilon A_{\gamma(x)}$,

$$\begin{aligned}\sum_{z\varepsilon\gamma^{-1}(A_{\gamma(x)})} r_{xz}\psi_{za}^{(n)}(t-u) &= \sum_{b\varepsilon A_{\gamma(x)}} \sum_{z\varepsilon\gamma^{-1}(b)} r_{xz}\psi_{za}^{(n)}(t-u) \\ &= \sum_{b\varepsilon A_{\gamma(x)}} \left(\sum_{z\varepsilon\gamma^{-1}(b)} r_{xz}\right) \psi_{ba}^{(n)}(t-u) \\ &= \sum_{b\varepsilon A_{\gamma(x)}} \psi_{ba}^{(n)}(t-u),\end{aligned}$$

thus $\psi_{xa}^{(n+1)}$ depends only on $\gamma(x)$. It follows that φ_{xa} depends only on $\gamma(x)$. Let $\varphi_{ba}(t)$ = the value of $\varphi_{xa}(t)$ on $\gamma^{-1}(b) = P\{\gamma(x_t) = a \mid \gamma(x_0) = b\}$. By differentiation we obtain

$$\frac{d}{dt}\varphi_{ba}(t) = \sum_{d\varepsilon B_b} \varphi_{da}(t) + \lambda \sum_{d\varepsilon A_b} \varphi_{da}(t) - [|b| + \lambda |A_b|]\varphi_{ba}(t),$$

whence it follows that the Markov process $\gamma(x_t)$ has the transition rate matrix Γ as stated in the theorem.

Theorem 2: The equilibrium distribution p of $\gamma(x_t)$ is given by

$$p_a = p_0\lambda^{|a|}, \quad p_0 = \left(\sum_{b\varepsilon\gamma(S)} \lambda^{|b|}\right)^{-1} = 1/\Phi(\lambda). \tag{1}$$

It satisfies the equation

$$(|a| + |A_a|)p_a = \sum_{b\varepsilon A_a} p_b + \lambda \sum_{b\varepsilon B_a} p_b, \qquad a \varepsilon (S). \tag{2}$$

Among all distributions q over $\gamma(S)$ satisfying the condition

$$\sum_{a\varepsilon\gamma(S)} q_a|a| = \sum_{a\varepsilon\gamma(S)} p_a|a|,$$

p maximizes the entropy functional

$$H = -\sum_{a \epsilon \gamma(S)} q_a \log q_a. \tag{3}$$

Proof: Equation (2) is the equilibrium condition for the matrix Γ of Theorem 1. The solution (1) can be verified by substitution, and the external property (3) of equation (1) is well known.

Remark 3: It can be seen that $\gamma(x_t)$ is the same stochastic process as would be obtained by applying the so-called "thermodynamic" model[6] proposed by the author to the state space $X = \gamma(S)$. This fact is the heuristic reason behind Theorem 2. As a result, all the special features of the "thermodynamic" model are present here, with $\Phi(\cdot)$ playing the role of the partition function.

Remark 4: $\gamma(x_t)$ is a reversible process. Its rate matrix is symmetrizable—that is, a symmetric operator in the space with inner product $\sum_{a \epsilon \gamma(S)} s_a t_a p_a$. From this fact follow useful inequalities for the covariance of $\gamma(x_t)$, as noted in an earlier work.[6] These inequalities bave application to sampling error in traffic measurements.

Corollary: The carried load $E \mid \gamma(x_t) \mid (= E \mid x_t \mid)$ *is given by*

$$E|x_t| = \lambda \frac{d}{d\lambda} \log \Phi(\lambda),$$

and the variance of the load is

$$\left(\lambda^2 \frac{d^2}{d\lambda} + \lambda \frac{d}{d\lambda}\right) \log \Phi - \lambda^2 \left(\frac{d}{d\lambda} \log \Phi\right)^2.$$

Theorem 3: If $\alpha_x = \alpha_{|x|}$ *, then the probability of blocking is given by*

$$1 - \frac{1}{\lambda} \frac{\sum_{a \epsilon \gamma(S)} |a| \lambda^{|a|}}{\sum_{a \epsilon \gamma(S)} \alpha_{|a|} \lambda^{|a|}}.$$

Proof: This proof comes directly from Theorem 2 and Ref. 1, since only completely blocked calls are rejected. Note that in Ref. 1, β counts the blocked calls, whereas here it counts only the completely blocked.

Remark 5: If the network is two-sided with N terminals on a side, then $\alpha_x = (N - \mid x \mid)^2$ and the loss is

$$1 - \frac{\frac{d}{d\lambda} \Phi}{\left[N^2 - (2N-1)\lambda \frac{d}{d\lambda} + \lambda^2 \frac{d^2}{d\lambda^2}\right] \Phi}.$$

If it is one-sided, with T terminals, then $\alpha_x = \binom{T-2|x|}{2}$, and the loss is

$$1 - \frac{2\frac{d}{d\lambda}\Phi}{\left[T^2 - T - (4T-6)\lambda\frac{d}{d\lambda} + 4\lambda^2\frac{d^2}{d\lambda^2}\right]\Phi}.$$

V. APPLICATION: CONCENTRATING OUTER SWITCHES

Let us consider a connecting network of the familiar type used in recent studies[1,4] and depicted in Fig. 1. For our purposes the $r \times r$ networks in the middle stage might themselves be multi-stage networks; we shall require only that they be rearrangeable, so that if $n \leq m$ the whole network itself is rearrangeable. However, we are interested in the case $n > m$ of "concentrating outer switches". We pose and answer the question what is the probability of blocking if $n > m$ and if we follow a policy of complete rearrangement, that is, if the existing calls are assigned new routes whenever this is necessary to accommodate a new, not completely blocked, call. The blocking is of course due entirely to the concentrators. By Theorem 3 and the remark following it, it suffices to calculate the partition function.

It is evident that to calculate the partition function $\Phi(\lambda)$ it is enough to know $|L_k|$, the cardinality of the set L_k of realizable assignments in which k inlets are busy, for $k \geq 0$. Since the network can be obtained from a rearrangeable one by substituting concentrators for square

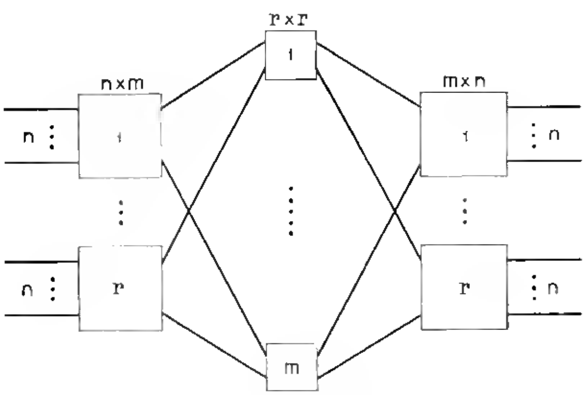

Fig. 1—A connecting network.

TABLE I—VALUES OF $\psi(r, k)$ FOR $n = 3$ AND $m = 2$

r \ k	0	1	2	3	4	5	6	7	8	9
1	1	3	3	0	0	0	0	0	0	0
2	1	6	15	18	9	0	0	0	0	0
3	1	9	36	81	108	81	27	0	0	0
4	1	12	66	216	459	648	594	324	81	0

outer switches, the realizable assignments will be precisely those involving m or fewer busy terminals on each outer switch. With m and n fixed, let $\psi(r, k)$ be the number of ways of choosing k inlet terminals from among the nr available, so that no more than m are from any one inlet switch. Let $i_1, \cdots, i_k$ be such a choice of inlets, and let $o_1, \cdots, o_k$ be a similar choice of outlets, also feasible in $\psi(r, k)$ ways. These chosen inlets can be mapped into the chosen outlets in $k!$ ways; each of these assignments will be realizable, and no others involving exactly k terminals will be. Hence in this case,

$$| L_k | = \psi^2(r, k)k!$$

To obtain a recurrence relation for $\psi(r, k)$, let us calculate $\psi(r+1, k)$ in terms of $\psi(r, j)$, $0 \leqq j \leqq k$. In this case we have one more outer switch available, and it can be seen that there are exactly $m + 1$ ways of using it: With no calls on it, with one call, with two, and so on up to m calls on it. In the first instance there are k on the other r switches, choosable in $\psi(r, k)$ ways; in the second there are $k - 1$ on the other r switches choosable in $\psi(r, k - 1)$ ways, and so on, up to m. If the $(r + 1)$th switch is to have j calls, these can be chosen in $\binom{n}{j}$ ways, but j must not exceed m. Thus

$$\psi(r + 1, k) = \sum_{j=0}^{m} \binom{n}{j} \psi(r, k - j), \qquad 0 \leqq k \leqq (r + 1)m.$$

It can be seen that $\psi(1, k) = \binom{n}{k}$ for $0 \leqq k \leqq m$. Introducing the generating function

$$\Psi_r(x) = \sum_{k=0}^{mr} x^k \psi(r, k),$$

we find at once that

$$\Psi_r(x) = \left(\sum_{j=0}^{m} \binom{n}{j} x^j \right)^r = \sum_{k=0}^{nr} x^k \psi(r, k).$$

We recall that if $A(x) = \Sigma a_n x^n$, a_n real, then

$$\Sigma\, a_n^2 x^n = \frac{1}{2\pi} \int_{-\pi}^{\pi} A(xe^{i\theta}) A(e^{-i\theta})\, d\theta.$$

To calculate $\Phi(\lambda)$, finally, we note that

$$\begin{aligned}
\Phi(\lambda) &= \sum_{a \epsilon A} \lambda^{|a|} \\
&= \sum_{k=0}^{mr} \lambda^k |L_k| \\
&= \sum_{k=0}^{mr} \lambda^k \psi^2(r, k) k! \\
&= \sum_{k=0}^{mr} \lambda^k \psi^2(r, k) \int_0^\infty e^{-u} u^k\, du \\
&= \int_0^\infty \sum_{k=0}^{mr} (\lambda u)^k \psi^2(r, k)\, du, \\
&= \frac{1}{2\pi} \int_0^\infty \int_{-\pi}^{\pi} \left(\sum_{j=0}^{m} \binom{n}{j} (u\lambda e^{i\theta})^j \right)^r \left(\sum_{j=0}^{m} \binom{n}{j} e^{-ij\theta} \right)^r d\theta\, du, \\
&= \frac{1}{2\pi} \int_0^\infty \int_{-\pi}^{\pi} Q^r(u\lambda e^{i\theta}) Q^r(e^{-i\theta})\, d\theta\, du,
\end{aligned}$$

where $i = (-1)^{\frac{1}{2}}$ and

$$Q(x) = \sum_{j=0}^{m} \binom{n}{j} x^j$$

is the generating function of the binomial coefficients $\binom{m}{j}$ truncated at m. This formula expresses $\Phi(\cdot)$ in terms of known polynomials and constitutes a complete solution of the problem posed, since all interesting quantities can be obtained from the partition function $\Phi(\cdot)$.

For small values of m and n, the recurrence for $\psi(r, k)$ is easily run out to give numerical answers. In Tables I and II we give some values of $\psi(r, k)$ and $|L_k|$ for $n = 3$, $m = 2$, and $r = 1, \cdots, 4$.

TABLE II—VALUES OF L_k FOR $n = 3$ AND $m = 2$

r \ k	0	1	2	3	4	5
1	1	9	18	0	0	0
2	1	36	450	1944	1944	0
3	1	81	2592	10566		
4	1	144	8712			

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